MATH 271.1 Operational Risk Project

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1 Introduction

Operational risk, although long recognized by banks and financial institutions as a potential source of loss, is a relatively young area of study that was only studied as an independent field for 20 years. In 2001, the Basel Committee on Banking Supervision released a series of papers that recognized operational risk as an important factor in shaping the risk profile of banks and other financial institutions. A consultative package on the new Basel Capital Accord (2001) was proposed that drew attention to the need of managers to take into consideration operational risks in financial institutions, and the need to have a minimum capital requirement that would be sufficient to deal with said underlying risks. In this document, operational risk was defined as: "the risk of direct or indirect loss resulting from inadequate or failed internal processes, people and systems or from external events."

The Basel II Accord is based on 3 pillars: (1) Minimum Capital Requirement, (2) Supervisory Review of Institution's Capital Adequacy, and (3) Market Discipline through Public Disclosure of Various Financial and Risk Indicators. Minimum Capital Requirements play the most crucial role in Basel II and requires the bank to maintain a minimum regulatory capital calculated for the three major risks: credit risk, market risk, and operational risk.

In this paper, we took a look at the data set containing loss amounts due to Damages to Physical Assets under the Asset Management business line and identify an appropriate risk capital charge for this particular line. The main approaches for measuring operational risks capital charge are the Basic Indicator Approach, Standardized Approach, Internal Measurement Approach, the Scorecard Approach, and the Loss Distribution Approach. In this paper, we place focus on using the Loss Distribution Approach to obtain the appropriate risk capital charge based on the data set. The best fit severity distribution among the Burr, Exponential, Gamma, Lognormal, Pareto, and Weibull was obtained using the Maximum Likelihood estimates for their parameters.

Furthermore, the best fit distribution for the frequency of the events was chosen among the $(a, b, 1)$ class of distributions - which include the Poisson, Binomial, Negative Binomial, Geometric distributions, as well as the Zero Truncated and Zero Modified versions of these distributions, if applicable. Similar to the severity distribution, the best parameters for these distributions will be obtained using the Maximum Likelihood estimate.

Lastly, using the Fast Fourier Transform, the discretized distribution of the Aggregate Loss Variable is obtained. With this, the 99% Value-at-Risk as well as the 99% Tail Value-at-Risk were calculated.

2 Data

The data set used in this paper contains the loss amounts due to Damages to Physical Assets under the Asset Management business line from January 2000 to December 2020. Over said period, 82 losses arising from damage to physical assets were recorded. The data set was cleaned by first rescaling the losses to the amount in millions to avoid any numerical stability problems when fitting a distribution. Figure 1 below shows the severity distribution of an event.

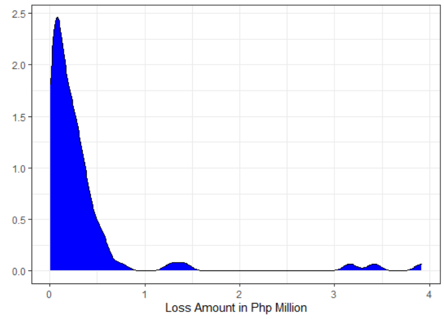

Figure 1: Density Plot of the Severity of Events

Moreover, a time interval of one (1) year was chosen for the frequency of the events. The frequency of events can be seen Figure 2 below.

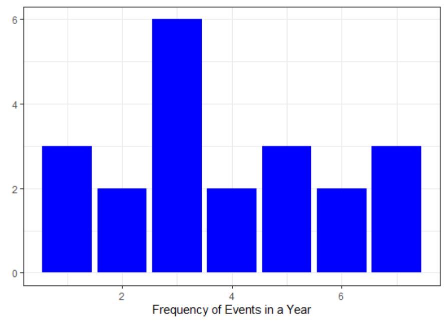

Figure 2: Bar Chart of the Frequency of Events

3 Severity Distribution

3.1 Fitting the Severity Distribution

For the severity of the losses, the loss amounts were rescaled to amount in millions. The values were then fit to the different severity distributions namely the Burr, Exponential, Gamma, Lognormal, Pareto, and Weibull. This was done using the `fitdistplus` package of R.

The parameters estimate was obtained using the Maximum Likelihood Estimates. If n observations $x_1, x_2, \ldots, x_n$ come from the same probability density function $f(x; \boldsymbol{\theta})$, the Maximum Likelihood Estimate $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ are the set parameters that would maximize the likelihood function

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i; \boldsymbol{\theta}).$$

3.1.1 Burr Distribution

The Maximum Likelihood estimate for the Burr Distribution are the following:

	Estimate	Std. Error
α	1.154641	0.4708952
γ	1.353386	0.2108058
θ	5.570330	2.6115792

Table 1: MLE Estimates for the Burr Distribution

3.1.2 Exponential Distribution

The Maximum Likelihood estimate for the Exponential Distribution are the following:

	Estimate	Std. Error
λ	2.867118	0.3185686

Table 2: MLE Estimates for the Exponential Distribution

3.1.3 Gamma Distribution

The Maximum Likelihood estimate for the Gamma Distribution are the following:

	Estimate	Std. Error
θ	0.481742	0.3888568
α	0.724027	0.0972481

Table 3: MLE Estimates for the Gamma Distribution

3.1.4 Lognormal Distribution

The Maximum Likelihood estimate for the Lognormal Distribution are the following:

	Estimate	Std. Error
μ	-1.884126	0.1395656
σ	1.256090	0.0986875

Table 4: MLE Estimates for the Lognormal Distribution

3.1.5 Pareto Distribution

The Maximum Likelihood estimate for the Pareto Distribution are the following:

	Estimate	Std. Error
α	2.336742	0.7614611
θ	0.451950	0.1977023

Table 5: MLE Estimates for the Pareto Distribution

3.1.6 Weibull Distribution

The Maximum Likelihood estimate for the Weibull Distribution are the following:

	Estimate	Std. Error
τ	0.771439	0.0597669
θ	0.286255	0.0437728

Table 6: MLE Estimates for the Weibull Distribution

3.2 Best Fit Severity Distribution

After obtaining the estimates for the parameters, the fit of the models are tested with respect to the loss amounts in the data set. The Kolmogorov–Smirnov statistic measures the maximum deviation between the cumulative distribution function of the model and the sample data set. The KS statistic is given by

$$\mathrm{KS} = \max |F(x) - F_n(x)|,$$

where $F(x)$ is the cumulative distribution function of the model distribution and $F_n(x)$ is the empirical cumulative distribution function obtained from the dataset given by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(x_i \leq x).$$

The model with the lowest KS statistic is the one in which the maximum difference between the empirical and model cumulative distribution function is the least. Thus, it is taken as the model that best fits the data set.

The Akaike Information Criterion was also used to evaluate which model had the best fit distribution. The Akaike Information Criteria can be computed by

$$\mathrm{AIC} = -2\ln(L) + 2k,$$

where L is the value of the likelihood function evaluated at parameter estimates and k is the number of estimated parameters.

Table 7 below shows the summary of the fitted severity distributions. Since the Lognormal distribution has the lowest KS Statistic and AIC, it will be the severity distribution used.

Distribution	KS Statistic	AIC
Burr	0.076265	-32.65468
Exponential	0.157708	-6.63577
Gamma	0.146766	-10.96864
Lognormal	0.073230	-34.42381
Pareto	0.073864	-30.83833
Weibull	0.118628	-18.05288

Table 7: Summary of Fitted Severity Distributions

4 Frequency Distribution

4.1 Fitting the Frequency Distribution

A fixed time interval of one (1) calendar year was chosen in order to get the frequency of the loss events. The values were then fit to distributions among the $(a, b, 1)$ class of distributions including Poisson, Binomial, Negative Binomial, Geometric distributions, as well as the Zero Truncated and Zero Modified versions of these distributions, if possible. We note that some of these distributions were not considered due to numerical instability and thus, the `fitdist` function was not able to obtain the MLE parameters for these distributions.

Similar to the severity distribution, the Maximum Likelihood Estimates were also used to set the parameters for these distributions. The Maximum Likelihood Estimate $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ are the set parameters that would maximize the likelihood function

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i; \boldsymbol{\theta}).$$

4.1.1 Poisson Distribution

The Maximum Likelihood estimate for the Poisson Distribution are the following:

		Estimate	Std. Error
Poisson	λ	3.857143	0.4285714
ZT Poisson	λ	3.768055	0.4386686
ZM Poisson	λ	3.767974	-
	p_0^M	0.000000	-

Table 8: MLE Estimates for the Poisson Distribution

4.1.2 Binomial Distribution

The Maximum Likelihood estimate for the Binomial Distribution are the following:

		Estimate	Std. Error
Binomial	n	7	-
	π	0.551020	-
ZT Binomial	n	7	-
	π	0.548927	-

Table 9: MLE Estimates for the Binomial Distribution

4.1.3 Negative Binomial Distribution

The Maximum Likelihood estimate for the Negative Binomial Distribution are the following:

		Estimate	Std. Error
Negative Binomial	r	415070	-
	π	0.000009	-
ZT Negative Binomial	r	42	156.5472831
	π	0.918018	0.2792734

Table 10: MLE Estimates for the Negative Binomial Distribution

4.1.4 Geometric Distribution

The Maximum Likelihood estimate for the Poisson Distribution are the following:

		Estimate	Std. Error
Geometric	π	0.205882	0.0400354
ZT Geometric	π	0.259260	0.0486915
ZM Geometric	π	0.259221	-
	p_0^M	0.000000	-

Table 11: MLE Estimates for the Geometric Distribution

4.2 Best Fit Frequency Distribution

Similar to the frequency distribution, the Akaike Information Criterion was also used to evaluate which model had the best fit distribtuion. The Akaike Information Criteria can be computed with

$$\text{AIC} = -2\ln(L) + 2k,$$

where L is the value of the likelihood function evaluated at parameter estimates and k is the number of estimated parameters.

Furthermore, we also used the Chi-Squared Goodness of Fit test to test the null hypothesis that the data set does indeed come from the distribution. The test statistic is computed by

$$\chi^2 = \sum_{j=1}^{k} \frac{(E_j - O_j)^2}{E_j},$$

where E_j is the number of expected observations and O_j is the number of observations.

Distribution	χ^2	p-value	AIC
Poisson	1.684361	0.4307702	88.49071
ZT Poisson	1.653532	0.4374617	87.55315
ZM Poisson	1.653619	0.1984671	89.55315
Binomial	7.919121	0.0048915	99.46210
ZT Binomial	8.050165	0.0045500	99.30482
Negative Binomial	1.684174	0.1943708	90.49072
ZT Negative Binomial	1.548877	0.2133012	89.49986
Geometric	10.862690	0.0043772	105.72370
ZT Geometric	5.516955	0.0633882	94.70947
ZM Geometric	5.516237	0.0188408	96.70947

Table 12: Summary of Fitted Frequency Distributions

Table 12 shows the summary of the fitted frequency distributions. Since the Zero Truncated Poission distribution has the lowest AIC and a p-value > 0.05, it will be the frequency distribution used.

5 Aggregate Loss Distribution

After determining the best fit distributions of the frequency and severity random variables, we can now solve for the probability density function of the aggregate loss distribution using the Fourier Transform Algorithm.

First, using the central difference approximation, the severity distribution is discretized as follows. Let $\delta > 0$. The discretization of the distribution $F_x(x)$ on $0, 1\delta, 2\delta, \ldots, (M-1)\delta$ under the central difference approximation is given by

$$f_k = \begin{cases} F_X\left(k\delta + \dfrac{\delta}{2}\right) - F_X\left(k\delta - \dfrac{\delta}{2}\right) & \text{if } k = 1, 2, \ldots \\ F_X\left(\dfrac{\delta}{2}\right) & \text{if } k = 0 \end{cases}.$$

Here, $M = 1024$ was used since an M that is a power of 2 must be used for the Fast Fourier Transform Algorithm. The plot of the discretized distribution of the Lognormal distribution with parameters $\mu = -1.884126$ and $\sigma = 1.256090$ can be seen in Figure 3 below.

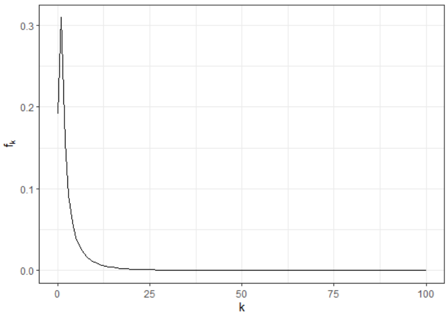

Figure 3: Discretized Severity Distribution

Afterwards, the Fast Fourier Transform is applied to obtain $\varphi_X(z)$, the characteristic function of the discretized distribution. The Fast Fourier Transform is the mapping

$$\begin{aligned}\tilde{f}_k &= \sum_{j=0}^{M-1} f_j \exp\left(\frac{2\pi i}{M} jk\right) \\ &= \sum_{j=0}^{\frac{M}{2}-1} f_{2j} \exp\left(\frac{2\pi i}{M/2} jk\right) + \exp\left(\frac{2\pi i}{M/2} k\right) \sum_{j=0}^{\frac{M}{2}-1} f_{2j+1} \exp\left(\frac{2\pi i}{M/2} jk\right).\end{aligned}$$

In contrast to the Discrete Fourier Transform Algorithm with a complexity of $O(M^2)$, the Fast Fourier Transform is able to compute the Discrete Fourier Transform in just $O(M \log_2 M)$. This is due to the latter using a divide-and-conquer technique which breaks down the problem into two sub-problems of the same nature.

Furthermore, if S is the aggregate loss random variable, we have

$$\varphi_S(z) = P_N\left(\varphi_X(z)\right),$$

where $P_N(z)$ is the probability generating function of the frequency distribution. Since we were able to identify that the best fit distribution of the frequency to be the Zero Truncated Poisson, we obtain the probability generating function $P_N(z)$ which is given by

$$\begin{aligned}P_N(z) &= \mathbb{E}(z^N) \\ &= \sum_{k=1}^{\infty} z^k p_k^T \\ &= \sum_{k=1}^{\infty} z^k \frac{e^{-\lambda}\lambda^k}{k!} \frac{1}{1-e^{-\lambda}} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left[\sum_{k=1}^{\infty} \frac{(z\lambda)^k}{k!}\right] \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left[\sum_{k=0}^{\infty} \frac{(z\lambda)^k}{k!} - 1\right] \\ &= \frac{1}{e^{\lambda}-1} \left[e^{\lambda z} - 1\right] \\ &= \frac{e^{\lambda z}-1}{e^{\lambda}-1}.\end{aligned}$$

From these equations, the characteristic function $\varphi_S(z)$ is obtained. Applying the inverse of the fast Fourier Transform, the discretized distribution of S is obtained with the same complexity.

$$f_k = \frac{1}{M}\sum_{j=0}^{M-1}\tilde{f}_j \exp\left(-\frac{2\pi i}{M}jk\right).$$

The probability density function and cumulative density function of the aggregate loss variable are shown below.

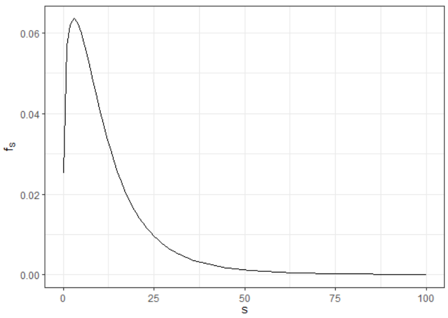

Figure 4: Probability Density Function of S

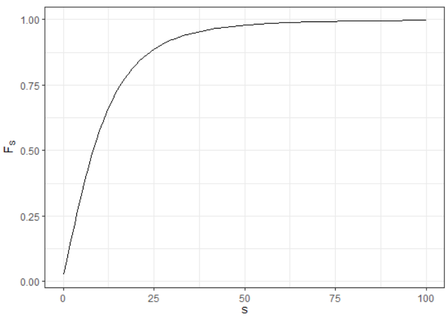

Figure 5: Cumulative Distribution Function of S

6 99% VaR and 99% TVaR

After obtaining the best fit distribution for the aggregate loss random variable, we proceed to solve for the 99% Value-at-Risk denoted by $\text{VaR}_{0.99}(S)$. The Value-at-Risk is defined as

$$\text{VaR}_\alpha(S) = \inf\{s_\alpha \in \mathbb{R} : P(S > s_\alpha) \leq \alpha\}.$$

The obtained 99% Value-at-Risk was Php 6,616,886.

Furthermore, the Tail Value-at-Risk was also calculated. The Tail Value-at-Risk at a given confidence level α is the expected loss given that the loss exceeds $s_\alpha = \text{Var}_\alpha(S)$. That is, the TVaR is given by

$$\begin{aligned}\text{TVaR}_\alpha(S) &= \mathbb{E}(S \mid S > s_\alpha)\\ &= \frac{1}{P(S > s_\alpha)}\sum_{s > s_\alpha} s f_S(s).\end{aligned}$$

The computed 99% Tail Value-at-Risk was Php 9,615,192.

7 Conclusion

This paper inspected the data set of the loss amounts due to Damages to Physical Assets under the Asset Management business line from January 2000 to December 2020. After fitting several distributions using the `fitdistplus` package of R, the best fit severity random variable was found to be a Lognormal distribution with $\mu = -1.884126$ and $\sigma = 1.256090$. Moreover, the best fit frequency random variable was a Zero Truncated Poisson Distribution with $\lambda = 3.768055$.

Furthermore, using the Fast Fourier Transform Algorithm, the probability density function and cumulative distribution function of the aggregate loss random variable was obtained. Using the results, we obtain a 99% Value-at-Risk of Php 6,616,886 and a 99% Tail Value-at-Risk of Php 9,615,192.

Overall, the results show that Operational Risk losses indeed are significant potential source of loss for banks and financial institutions and must be studied and modeled so as to be able to allocate sufficient capital buffer. Without a sufficient risk capital charge, these institutions would likely be unable or unprepared to deal with these losses. The results show that at a 99% level of confidence, a capital charge of Php 6.6 million would be sufficient to cover the total operational loss for this particular business segment and event type. However, in the event that loss exceeds Php 6.6 million, the expected loss amount is Php 9.6 million.

References

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